# THE ROLE OF EXPECTATIONS IN AN ADAPTIVE SEARCH MODEL*

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*Resumen:* El propósito de este artículo es encontrar condiciones suficientes bajo las cuales los resultados del modelo simple de búsqueda se mantengan después de introducir un proceso de aprendizaje. Se demuestra que estas propiedades se mantienen bajo supuestos más bien simples. De manera intuitiva, lo que se necesita es que el agente no tenga expectativas "explosivas".

*Abstract:* The purpose of this paper is to find sufficient conditions under which the results of the simple search model are preserved when learning is allowed. It is shown that the properties of the simple search model are preserved with very simple assumptions. Intuitively, we need a "non-explosive" agent. That is, expectations must be smooth.

## 1. Introduction

The simple search model has been useful to analyze different problems in many areas of economics (for example, see Lippman and McCall, 1976, and 1993). However, the stationary assumption (in the simple search model all time dependent variables are held fixed) is too strong for some applications.

Several efforts have been made to study the behavior of a searcher under a non-stationary environment, in particular, to obtain condi-

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tions under which the results of this model are the same than those of the simple one (Rothschild, 1974; Rosenfield and Shapiro, 1981, and Bikhchandani and Sharma, 1990). These authors have shown that if the search process follows some specific rules (a specific updating process, or a particular offer distribution) the results of the non-stationary model replicate those of the simple one. The objective of this paper is to show the role of expectations in this case. The results of stationary search models are preserved if it is assumed that agents have "non-explosive" expectations. Also, a very simple (practically naive) updating process, compatible with this required characteristic of expectations, is presented.

The paper is organized as follows. In the next section the search process is presented. In section 3 the non-stationary search model is developed. Here I find out the conditions under which the results of the simple model are preserved. Conclusions close the paper.

## 2. The Search Process

To describe search I will use the R&D investment decision. The reason for doing this is that here a non-stationary approach is more adequate, R&D is basically a learning process. The firm that invests in R&D learns about new products or processes. Therefore, it seems reasonable to assume that it follows a search process with learning, i.e., a non-stationary.

The objective of the searcher, a firm in this case, is to obtain the highest value of an innovation. To achieve its objective the firm invests dollars, a fixed amount, in an R&D project at the beginning of each period. At the end of it the firm knows with certainty the "value" of the new (improved) good or process, which is called $p_t$.[1]

The "value" $p_t$ is the net present value of the future cash flows associated with the technology at its current stage.[2] At all periods the firm observes the current offer $p_t$.

[1] In a different approach the firm does not know with certainty the value of the project. See Lippman and McCardle (1991).

[2] The NPV is net with respect to the production costs of the new (improved) good or the variable costs of the new (improved) process, not net with respect to the R&D costs.

It is assumed that the firm can adopt any innovation at zero cost. This implies that the firm adopts all innovations with a value higher than the one of the current technology. Then $\xi_t$ is defined as the best offer after $t$ periods or, equivalently, the value of the technology being used. Note that $\xi_t$ is no -decreasing.[3] If $p_t \geq \xi_{t-1}$ the firm adopts the improved technology at its current stage. However, to decide whether to continue the project, the firm compares the best value of the innovation so far ($\xi_t$) with the expected net value after investing (searching) another period. Thus, there are two comparisons to be made; $p_t$ against $\xi_t$ and the expected value of search against $\xi_t$. Every period in which the value of the innovation is "large enough" the new (or improved) product or process is adopted. However, only when $\xi_t$ is larger than the expected value of search the firm stops the R&D project. As noted elsewhere, in the stationary case this search policy exhibits the reservation value and myopic properties.[4]

The reservation value property means that to decide whether to stop or not the searcher compares her options with a benchmark value. The myopic property, on the other hand, means that the best value is compared with the expected value of the innovation in the next period, net of search costs. That is, the searcher behaves as if only one period remains.

### 3. Adaptive Search

As stated above, the simple search model assumes that all time dependent variables are held fixed (the stationary assumption). In this section I will show that if we relax this assumption there are no effects on the predictions of the model, if some restrictions are .added. Other authors (Rothschild, 1974; Rosenfield and Shapiro, 1981, and Bikhchandani and Sharma, 1990) reached at the same conclusion. This paper adds to their results in the following sense: even with very simple restrictions

[3] If $p_{t+1} > \xi_t$ then $\xi_{t+1} = p_{t+1}$, and if $p_{t+1} \leq \xi_t$, then $\xi_{t+1} = \xi_t$, therefore $\xi_{t+1} \geq \xi_t$.

[4] Lippman and McCall (1976) and Reinganum (1982), among others, show that when recall is not allowed these properties hold. It is well known that in the stationary case the recall assumption is innocuous, therefore in the recall case the properties mentioned in the text hold as well.

(almost naive) the results of the simple search model are preserved. This means that it is robust enough to the attempts made to construct more "realistic" models.

Following Rosenfield and Shapiro (1981), the crucial element to prove is that the adaptive search model must have the reservation value and myopic properties. If these two characteristics of the simple model are preserved, the comparative static analysis follows through.

In the stationary case it is assumed that the searcher knows the true distribution function of the offers, and it is fixed. Here it is assumed that the firm does not know the true offer's distribution function. However, it holds some beliefs that are updated every period. Also, it is assumed that the true distribution function is constant over the R&D process. That is, the value of innovation in period $t$, $P_t$, is a non-negative random variable with unknown cumulative density function. The searcher has some beliefs about the distribution in $t$, $F_t$, and the latter is updated every period. This assumption implies that the relevant distribution, the one that the searcher uses, is not constant anymore. Therefore, the search process is non-stationary.

It is assumed that $F_t$ depends on the previous offers, i.e., $F_t = F(p_t \mid \rho)$, with p being the vector of previous offers, $\rho = (p_1, p_2, ..., p_{t-1})$. Therefore, the searcher updates its prior after each draw. The assumed updating process is quite simple: every time the current offer is better (worse) than the value of the in-use technology, the whole probability function moves to the right (left). That is, if the searcher observes an offer below $\xi_t$ at $t+1$, then the posterior distribution is dominated by the prior stochastically in first order. This implies that the expected value of search decreases every time a worse offer is observed. On the other hand, if the searcher observes an offer higher than $\xi_t$ at $t+1$, then the posterior distribution dominates the prior stochastically in first order. Then the expected value of search goes up (this is in the spirit of Bikhchandani and Sharma, 1990).

In order to have a finite search process, it is necessary to assume that the expected value of search increases less than the value of the in-use technology ($\xi_t$), at least after some time, whenever a good draw occurred. That is, it can be the case that at the beginning the expected value of search increases mores than $\xi_t$. However, as time goes by the expected value of search must increase less than $\xi_t$, to have a finite process. Note that there is no requirement for the bad draws. In this case $\xi_t$ does not change and the expected value of search decreases. Thus, the probability of stopping increases.

With the structure already given, it is possible to state the problem more formally. At period $t$ the firm has to decide whether to continue its R&D project or not, when the best value of the innovation is $\xi_t$. Here the Bellman equation that defines the problem is

$$(\xi_t, \rho) = \max\{\xi_t, = -c + \beta\int_m^M V(p_{t+1}, \rho')f(p_{t+1} \mid \rho)dp_{t+1}, \qquad (1)$$

where $M$ is the highest and $m$ the lowest value that $p_{t+1}$ (the realization of $P_{t+1}$) can take, and $\rho'$ represents the vector with one more offer. Note that there are two state variables, the value of the in-use technology ($\xi_t$), and the vector of previous offers. Note also that the distribution function that the searcher uses includes the information given by the observation of the $t$-th offer.

I will first show that the optimal policy is myopic and then that it can be characterized by a reservation value. The difference with Rosenfield and Shapiro (1981) is the restriction imposed to guarantee these two results.

For simplicity I will call the second part of the integral in (1) as $\varphi_{t+1}$, the net expected value of search. That is,

$$\varphi_{t+1} = -c + \beta\int_m^M V(p_{t+1}, \rho')f(p_{t+1} \mid \rho)dp_{t+1}. \qquad (2)$$

Following Rosenfield and Shapiro (1981), the myopic property holds if the one period ahead looking policy is optimal. If only one period remains the searcher will stop (continue) if the best offer so far is larger (smaller) than the net expected value of searching the last period. That is, stop (continue) if the following holds:

$$_t \geq (<) -c + \beta\int_m^M \max(\xi_t, p_{t+1})f(p_{t+1} \mid \rho)dp_{t+1}, \qquad (3)$$

where in general, for $\infty$ periods remaining, the rule is to stop (continue) if the following inequality holds:

$$_t \geq (<) -c + \beta\int_m^M V(p_{t+1}, \rho')f(p_{t+1} \mid \rho)dp_{t+1} = \varphi_{t+1}. \qquad (4)$$

Rosenfield and Shapiro (1981) prove that it is sufficient to have the net expected value of search decreasing with extra observations, for the myopic property to hold. In Proposition 1 it is shown that if $\varphi_{t+1}$ increases proportionally less than $\xi_t$, then the myopic property holds. Therefore, this requirement is less stringent than the one demanded by Rosenfield and Shapiro.

PROPOSITION 1: *If* $\partial\varphi_{t+1} / \partial\xi_t < 1$ *then the myopic property holds.*

PROOF: Let's check the stopping rule first. Then assume that $\xi_t \geq \varphi_{t+1}$. If this is the case and given the hypothesis, $\xi_{t+1} \geq \varphi_{t+2}$ holds and therefore the searcher will also stop at $t+1$, with certainty. This implies that $V(p_{t+1}, \rho') = \max(\xi_t, p_{t+1})$, and therefore

$$
\begin{aligned}
{}_t \geq -c + \beta \int_m^M \max(\xi_t, p_{t+1}) f(p_{t+1} \mid \rho) dp_{t+1} = \\
-c + \beta \int_m^M V(p_{t+1}, \rho) f(p_{t+1} \mid \rho') dp_{t+1} \quad . \qquad (5)
\end{aligned}
$$

The first inequality is the one period ahead looking rule, which says "stop". The second one is the $\infty$ periods ahead looking rule, which also says "stop". Therefore, the myopic stopping rule is errorless.

To check the continuation policy, assume $\xi_t < \varphi_{t+1}$. Here the searcher will not stop at $t$, and maybe not even at $t+1$, therefore $V(p_{t+1}, \rho') \geq \max(\xi_t, p_{t+1})$. This implies

$$
\begin{aligned}
{}_t \leq -c + \beta \int_m^M \max(\xi_t, p_{t+1}) f(p_{t+1} \mid \rho) dp_{t+1} \\
\leq -c + \beta \int_m^M V(p_{t+1}, \rho') f(p_{t+1} \mid \rho) dp_{t+1} \qquad (6)
\end{aligned}
$$

Again, the first inequality corresponds to the one period rule, which says "continue", while the second one corresponds to the $\infty$ periods rule, which says also "continue". Therefore the myopic continuation rule is errorless and it is verified that the myopic policy is optimal. ■

The next step is to prove that with the same assumption about the relation between the expected value of search and the value of the in-use technology, i.e., $\partial\varphi_{t+1} / \partial\xi_t < 1$, the optimal policy is reservation.

PROPOSITION 2: *If* $\partial\varphi_{t+1}/\partial\xi_t < 1$ *then the optimal policy is reservation in terms of the best value so far*

PROOF: At the beginning of the search process $\xi_0 = 0$ and $\varphi_1 > 0$, then the first positive offer is accepted, thus making $\xi_1 = p_1$. Given that $\partial\varphi_{t+1}/\partial\xi_t < 1$, the best value so far and the expected value will cross only once. After that point the searcher will stop, the expected value ($\varphi_{t+1}$) will be smaller than the best offer so far ($\xi_t$). ■

Thus, the searcher compares this period offer with the best value so far. If the former is larger, then the offer is accepted. Also, if the best offer so far is larger than the newly computed expected value, the search process stops.

Therefore, I have shown that if the expected value of search increases proportionally less than the reservation value, the two desired properties are preserved. The searcher cannot explode in optimism (pessimism) after a better (worse) offer arrives, to have the desired properties. If this is the case, the simple search model is suitable to analyze R&D. Moreover, as seen before, this assumption also guarantees that the search process is finite.

Now it is time to show a case under which the condition $\partial\varphi_{t+1}/\partial\xi_t < 1$ holds. That is, to present an updating process compatible with a "non-explosive" agent. To do so it is necessary to be more precise on the updating process. It is assumed that if $p_t = \xi_{t-1} \pm s$, then $P_{t+1} = P_t \pm s\mu(t)$, where $P_t$ is the random variable at $t$. That is, the distribution moves at most by $s$. Also, to assure convergence it is assumed $\mu' < 0$, $\mu(0) = 1$ and $\mu(\infty) = 0$. That is, the searcher puts increasing weight in the prior, as time goes by.

PROPOSITION 3: *If the optimal policy is myopic and reservation, and if the updating process is one such that if* $p_t = \xi_{t-1} + s$ ($p_t = \xi_{t-1} - s$) *the searcher believes that the distribution moves to the right (left) at most by s, then* $\partial\varphi_{t+1}/\partial\xi_t < 1$ *holds.*

PROOF: Here it will be shown that the expected value of search changes less than the improvement of the reservation value (best offer), under the assumed updating process.

Given that the best offer is non-decreasing the relevant case is when $\xi_t = p_t = \xi_{t-1} + s$. Under the updating process described in the text $P_{t+1}$ dominates $P_t$ in first order. That is, $P_{t+1} = P_t + s\mu(t)$, with every possible realization $p_{t+1} = p_t + s\mu(t)$. By assuming the myopic and reservation properties, equation (2) becomes

$$\varphi_t = -c + \beta\xi_{t-1} F(\xi_{t-1} \mid \rho) + \beta \int_{\xi_{t-1}}^{M} p_t f(p_t \mid \rho) dp_t \tag{7}$$

and

$$\varphi_{t+1} = -c + \beta\xi_t F(\xi_t \mid \rho') + \beta \int_{\xi_t}^{M} p_{t+1} f(p_{t+1} \mid \rho') dp_{t+1}. \tag{8}$$

Equations (7) and (8) imply

$$\begin{aligned}\varphi_{t+1} &= -c + \beta\xi_{t-1} F(\xi_{t-1} \mid \rho) + \beta s F(\xi_t \mid \rho') \\ &+ \beta \int_{\xi_{t-1}}^{M} p_t f(p_t \mid \rho) dp_t + \beta s\mu(t)[1 - F(\xi_t \mid \rho')].\end{aligned} \tag{9}$$

These equations imply

$$\varphi_{t+1} = \varphi_t + \beta s[\mu(t) - \mu(t)\, F(\xi_t \mid \rho') + F(\xi_t \mid \rho')] < \varphi_t + s, \tag{10}$$

given that $\beta < 1$ and $[\mu - \mu F + F] < 1$.[5]

Therefore, in this case the expected value of search increases less than the improvement in the reservation value. That is, $\partial\varphi_{t+1} / \partial\xi_t < 1$. ■

Note that if a "worse" offer is observed $\xi_t$ is not affected, but $\varphi_{t+1}$ decreases. However, this does not affect the reservation value and myopic properties. Let $p_t = \xi_{t-1} - s$. Then, the value of the technology in use (the reservation value) does not change, $\xi_t = \xi_{t-1}$. Therefore,

$$\varphi_{t+1} = \varphi_t - \beta s\mu(t)[1 - F(\xi_t \mid \rho')] \geq \varphi_t - s, \tag{11}$$

[5] Note that $[\mu - \mu F + F] \leq 1 \Leftrightarrow F(1 - \mu) \leq 1 - \mu \Leftrightarrow F \leq 1$. On the other hand, $[\mu - \mu F + F] > 1 \Leftrightarrow F(1 - \mu) > 1 - \mu \Leftrightarrow F > 1$, wich is not feasible.

given that $\beta < 1$. Therefore, the expected value of search decreases less than the worsening of the offer.

Therefore, for any distribution function I have shown that it is possible to generate a "non-explosive" agent with a sensible updating process. Moreover, this special case seems a sensible process for R&D, because any time a "better" offer is observed the searcher believes the distribution is a "better" one. However, after several draws the information given by the actual observation is smaller (it may be only a lucky, or unlucky, draw), thus allowing for convergence. The updating process may appear "naive" as well. This is good, because it shows that it is not necessary to assume specific distribution functions and complicated updating processes to keep the reservation value property.

## 4. Conclusions

This paper shows that the properties of the simple search model are preserved with very simple assumptions. Intuitively, we need a "non-explosive" agent. That is, expectations must be smooth. Moreover, it was shown that this smoothness can be achieved with a very simple updating process.

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